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Adjoints of composition operators with rational symbol

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Abstract

Building on techniques developed by Cowen and Gallardo-Gutiérrez, we find a concrete formula for the adjoint of a composition operator with rational symbol acting on the Hardy space H^2. We consider some specific examples, comparing our formula with several results that were previously known.

Keywords: Composition operator; Adjoint; Hardy space

1. Preliminaries

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The *Hardy space* H^2 is the Hilbert space consisting of all analytic functions $f(z) = \sum a_n z^n$ on $\mathbb{D}$ such that

$$\|f\|_2 = \sqrt{\sum_{n=0}^{\infty} |a_n|^2} < \infty.$$

If $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ belong to H^2, the inner product $\langle f, g\rangle$ can be written in several ways. For example,

$$\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} = \lim_{r\uparrow 1} \int_0^{2\pi} f\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\,\frac{d\theta}{2\pi}.$$

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Any function f in H^2 can be extended to the boundary of $\mathbb{D}$ by means of radial limits; in particular, $f(\zeta)=\lim_{r\uparrow 1} f(r\zeta)$ exists for almost all ζ in $\partial\mathbb{D}$. (See Theorem 2.2 in [5].) Furthermore, we can write

$$\langle f,g\rangle=\int_0^{2\pi} f\big(e^{i\theta}\big)\overline{g\big(e^{i\theta}\big)}\,\frac{d\theta}{2\pi}=\frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\overline{g(\zeta)}\,\frac{d\zeta}{\zeta}.$$

It is often helpful to think of H^2 as a subspace of $L^2(\partial\mathbb{D})$. Taking the basis $\{z^n\}_{n=-\infty}^{\infty}$ for $L^2(\partial\mathbb{D})$, we can identify the Hardy space with the collection of functions whose Fourier coefficients vanish for $n\leqslant -1$.

One important property of H^2 is that it is a *reproducing kernel Hilbert space*. In other words, for any point w in $\mathbb{D}$ there is some function K_w in H^2 (known as a *reproducing kernel function*) such that $\langle f,K_w\rangle=f(w)$ for all f in H^2. In the case of the Hardy space, it is easy to see that $K_w(z)=1/(1-\bar{w}z)$.

At this point, we will introduce our principal object of study. Let φ be an analytic map that takes $\mathbb{D}$ into itself. The *composition operator* C_φ on H^2 is defined by the rule

$$C_\varphi(f)=f\circ\varphi.$$

It follows from Littlewood's Subordination Theorem (see Theorem 2.22 in [4]) that every such operator takes H^2 into itself. These operators have received a good deal of attention in recent years. Both [4] and [10] provide an overview of many of the results that are known.

2. Adjoints

One of the most fundamental questions relating to composition operators is how to obtain a reasonable representation for their adjoints. It is difficult to find a useful description for C_φ^*, apart from the elementary identity

$$C_\varphi^*(K_w)=K_{\varphi(w)}. \tag{1}$$

(See Theorem 1.4 in [4].) In 1988, Cowen [2] used this fact to establish the first major result pertaining to the adjoints of composition operators:

Theorem 1 *(Cowen). Let*

$$\varphi(z)=\frac{az+b}{cz+d}$$

be a nonconstant linear fractional map that takes $\mathbb{D}$ into itself. The adjoint C_φ^* can be written $T_gC_\sigma T_h^*$, for

$$g(z)=\frac{1}{-\bar{b}z+\bar{d}},\qquad \sigma(z)=\frac{\bar{a}z-\bar{c}}{-\bar{b}z+\bar{d}},\quad\text{and}\quad h(z)=cz+d,$$

where T_g and T_h denote the Toeplitz operators with symbols g and h, respectively.

While Cowen only stated this result for nonconstant φ, it is easy to see that the formula also holds for constant maps, provided that φ is written in the form

$$\varphi(z)=\frac{b}{d}=\frac{0z+b}{0z+d},$$

so that $\sigma(z)=0$. In that case, C_φ and C_σ can simply be considered point-evaluation functionals.

It is sometimes helpful to have a more concrete version of Cowen's adjoint formula. Recalling that T_z^* is the backward shift on H^2, we see that

$$\begin{aligned}\big(C_\varphi^*f\big)(z)&=\left(\frac{1}{-\bar{b}z+\bar{d}}\right)\left(\bar{c}\left(\frac{f(\sigma(z))-f(0)}{\sigma(z)}\right)+\bar{d}f\big(\sigma(z)\big)\right)\\&=\left(\frac{1}{-\bar{b}z+\bar{d}}\right)\left(\left(\frac{\bar{c}+\bar{d}\sigma(z)}{\sigma(z)}\right)f\big(\sigma(z)\big)-\frac{\bar{c}f(0)}{\sigma(z)}\right)\\&=\frac{(\overline{ad}-\overline{bc})z}{(\bar{a}z-\bar{c})(-\bar{b}z+\bar{d})}f\big(\sigma(z)\big)+\frac{\bar{c}f(0)}{\bar{c}-\bar{a}z}.\end{aligned}\tag{2}$$

A similar calculation appears in [7].

In recent years, numerous authors have made the observation that

$$\big(C_\varphi^* f\big)(w) = \langle f, K_w \circ \varphi\rangle = \int_0^{2\pi} \frac{f(e^{i\theta})}{1-\overline{\varphi(e^{i\theta})}w}\,\frac{d\theta}{2\pi}. \tag{3}$$

This fact seems particularly helpful when considering composition operators induced by rational maps. In an unpublished manuscript, Bourdon [1] uses it to find a representation for C_φ^* when φ belongs to a certain class of "quadratic fractional" maps. It is the principal tool used by Effinger-Dean, Johnson, Reed, and Shapiro [6] to calculate $\|C_\varphi\|$ when φ is a rational map satisfying a particular finiteness condition. Equation (3) is also the starting point from which both Martín and Vukotić [8] and Cowen and Gallardo-Gutiérrez [3] attempt to describe the adjoints of all composition operators with rational symbol. It is the content of this last paper that serves as the catalyst for our current discussion.

The results of Cowen and Gallardo-Gutiérrez are stated in terms of *multiple-valued weighted composition operators*. Suppose that ψ and σ are a compatible pair of multiple-valued analytic maps on $\mathbb{D}$ (in a sense the authors describe in their paper), with $\sigma(\mathbb{D}) \subseteq \mathbb{D}$. The operator $W_{\psi,\sigma}$ is defined by the rule

$$(W_{\psi,\sigma} f)(z) = \sum \psi(z) f\big(\sigma(z)\big),$$

the sum being taken over all branches of the pair ψ and σ. Whenever we encounter such an operator in this paper, the function ψ will actually be defined in terms of σ.

Before considering their adjoint theorem, we need to remind the reader of a particular piece of notation. If f is a (possibly multiple-valued) function acting on a subset U of the Riemann sphere, we define the function $\tilde{f}$ on the set $\{z \in \mathbb{C} \cup \{\infty\}\colon\ 1/\bar{z} \in U\}$ by the rule

$$\tilde{f}(z) = \overline{f\left(\frac{1}{\bar{z}}\right)}. \tag{4}$$

Cowen and Gallardo-Gutiérrez state their adjoint formula in terms of this notation:

Theorem 2 *(Cowen and Gallardo-Gutiérrez). Let φ be a nonconstant rational map that takes $\mathbb{D}$ into itself. The adjoint C_φ^* can be written $BW_{\psi,\sigma}$, where B denotes the backward shift operator and $W_{\psi,\sigma}$ is the multiple-valued weighted composition operator induced by $\sigma = 1/\widetilde{\varphi^{-1}}$ and $\psi = \widetilde{(\varphi^{-1})'}/\widetilde{\varphi^{-1}}$.*

Note that the function $\widetilde{(\varphi^{-1})'}$ in the numerator of ψ represents the "tilde transform" of $(\varphi^{-1})'$, as defined in line (4), rather than the derivative of $\widetilde{\varphi^{-1}}$. It is clear from the context of this theorem that the authors consider both B and $W_{\psi,\sigma}$ to be operators from H^2 into itself.

As we shall see, Theorem 2 is not actually correct in all cases. We will begin by considering whether this result is valid for linear fractional maps.

3. Linear fractional examples

If Theorem 2 were to hold in general, it would certainly have to agree with Theorem 1 in the case of linear fractional maps. We shall show that these two theorems rarely yield the same result. Let

$$\varphi(z) = \frac{az+b}{cz+d}$$

be a nonconstant map that takes $\mathbb{D}$ into itself. Note that

$$\varphi^{-1}(z) = \frac{dz-b}{-cz+a}.$$

Using the notation of Theorem 2, we can write

$$\widetilde{\varphi^{-1}}(z) = \overline{\varphi^{-1}(1/\bar{z})} = \frac{-\bar{b}z+\bar{d}}{\bar{a}z-\bar{c}},$$

and thus

$$\sigma(z) = 1/\widetilde{\varphi^{-1}}(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}}.$$

Also note that

$$\widetilde{\left(\varphi^{-1}\right)'}(z) = \overline{\left(\frac{ad - bc}{(-\frac{c}{z} + a)^2}\right)} = \frac{(\overline{ad} - \overline{bc})z^2}{(\bar{a}z - \bar{c})^2},$$

from which it follows that

$$\psi(z) = \widetilde{\left(\varphi^{-1}\right)'}(z)/\widetilde{\varphi^{-1}}(z) = \frac{(\overline{ad} - \overline{bc})z^2}{(\bar{a}z - \bar{c})(-\bar{b}z + \bar{d})}.$$

According to Theorem 2, the operator C_φ^* can be written as the product $BW_{\psi,\sigma}$. This would allow us to write

$$\left(C_\varphi^* f\right)(z) = \frac{\psi(z) f(\sigma(z)) - \psi(0) f(\sigma(0))}{z} = \frac{(\overline{ad} - \overline{bc})z}{(\bar{a}z - \bar{c})(-\bar{b}z + \bar{d})} f\left(\sigma(z)\right). \tag{5}$$

It is clear that line (5) agrees with line (2) only in one special case: when $c = 0$. In other words, Theorem 2 is only valid for those linear fractional maps that are actually linear.

There is more than one reason that Theorem 2 fails in this situation. We will present a pair of examples, each demonstrating one difficulty with the theorem.

Example 1. Consider the map

$$\varphi(z) = \frac{2z}{z + 4},$$

which certainly takes $\mathbb{D}$ into itself. Theorem 2, as manifested in line (5), says that

$$\left(C_\varphi^* f\right)(z) = \left(\frac{2z}{2z - 1}\right) f\left(\sigma(z)\right). \tag{6}$$

This statement cannot be correct, since the operator on the right-hand side does not take H^2 into itself. In particular, if $f(\sigma(1/2)) = f(0) \neq 0$, expression (6) fails to be analytic in $\mathbb{D}$.

While we will provide a more detailed discussion at a later point, the problem here is that $W_{\psi,\sigma}$ does not take H^2 into itself. The proper way to correct this difficulty is to follow the backward shift with the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2. In this particular example, we would obtain

$$\left(C_\varphi^* f\right)(z) = \left(\frac{2z}{2z - 1}\right) f\left(\sigma(z)\right) + \frac{f(0)}{1 - 2z},$$

which agrees with the adjoint formula given by line (2).

There is some ambiguity in the statement of Theorem 2, in that the backward shift can sometimes be construed to include a projection. Nonetheless, this issue is more than a notational nicety, as the absence of the projection causes Corollary 3.9 in [3] to be incorrect. In particular, the kernel of C_φ^* includes not only those functions that are sent to 0 as a result of the backward shift, but also those that are sent to 0 due to the projection.

The next example illustrates a different problem with Theorem 2.

Example 2. Let

$$\varphi(z) = \frac{z}{2z + 4}.$$

According to Theorem 2, as explained at the start of this section, the adjoint should be given by the formula

$$\left(C_\varphi^* f\right)(z) = \left(\frac{z}{z - 2}\right) f\left(\sigma(z)\right). \tag{7}$$

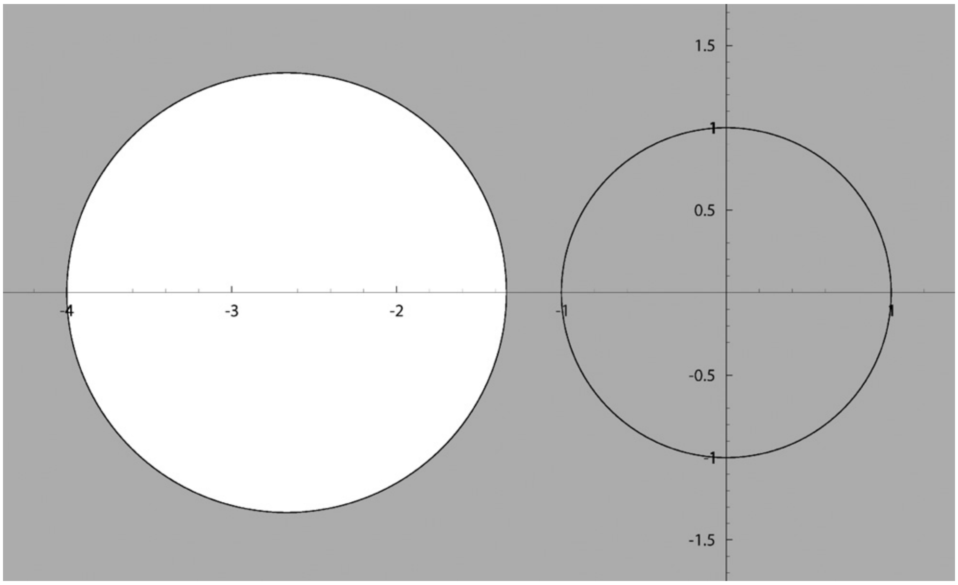

Fig. 1. The set $\varphi^{-1}(\mathbb{D})$ for $\varphi(z) = z/(2z+4)$, along with $\partial\mathbb{D}$.

Unlike the previous example, this expression always yields an analytic function on $\mathbb{D}$. Nevertheless, even if it did not contradict Theorem 1, Eq. (7) cannot provide a correct formula for C_φ^*. Since $\varphi(0) = 0$, line (1) tells us that C_φ^* must fix the kernel function $K_0(z) = 1$. It is obvious that the operator defined above does not fix constant functions.

The difficulty here is more subtle—and more fundamental—than that of the previous example. The proof of Theorem 2 (i.e. Theorem 3.8 in [3]) relies on the assumption that $\partial\varphi^{-1}(\mathbb{D})$, or at least one component thereof, is a simple closed curve enclosing $\partial\mathbb{D}$. In this instance, $\partial\varphi^{-1}(\mathbb{D})$ is a circle of radius $4/3$ centered at $z = -8/3$. In particular, $\partial\varphi^{-1}(\mathbb{D})$ does not enclose $\partial\mathbb{D}$. (See Fig. 1.) Consequently one cannot use Cauchy's theorem to show that

$$\int\limits_{\partial\mathbb{D}} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = \int\limits_{\partial\varphi^{-1}(\mathbb{D})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta}$$

for polynomials f and g, as asserted in the proof of Theorem 2, since the integrand has a distinct singularity within each circle (at $z = 0$ and $z = -2$, respectively). Without this purported equality, the remainder of the proof of Theorem 2 is invalid.

Note that $\partial\varphi^{-1}(\mathbb{D})$ need not even be a circle. For example, if we take

$$\varphi(z) = \frac{z}{z+4},$$

then $\partial\varphi^{-1}(\mathbb{D})$ is the vertical line $\operatorname{Re} z = -2$.

While these examples are particular to linear fractional maps, the problems they illustrate can manifest themselves with rational maps of any degree.

4. The main theorem

From this point onward, we will assume that φ is a rational map that takes $\mathbb{D}$ into itself. Our goal is to apply certain aspects of the proof of Theorem 2 (Theorem 3.8 in [3]) to obtain a general formula for C_φ^*. Before stating our main results, we need to extract an important calculation from the proof of Theorem 2.

Lemma 3 *(Cowen and Gallardo-Gutiérrez). Let φ be a nonconstant rational map that takes $\mathbb{D}$ into itself. Define the multiple-valued functions*

$$\sigma(z) = 1/\widetilde{\varphi^{-1}}(z) = 1/\overline{\varphi^{-1}(1/\bar{z})}$$

and

$$\psi(z) = \frac{z\sigma'(z)}{\sigma(z)}.$$

Suppose that $\partial\varphi^{-1}(\mathbb{D})$ is a bounded set. If f is a rational function with no poles on $\partial\mathbb{D}$ and g is a polynomial, then

$$\int_{\partial\varphi^{-1}(\mathbb{D})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = \int_{\partial\mathbb{D}} f(\zeta)\overline{\sum \psi(\zeta) g\big(\sigma(\zeta)\big)}\,\frac{d\zeta}{\zeta}. \tag{8}$$

In this expression, the function $\tilde{g}$ is defined as in line (4) *and the summation is taken over all branches of σ.*

While the map σ here is the same as in Theorem 2, the function ψ is slightly different. With a bit of work, the ψ from Theorem 2 can be rewritten $z^2\sigma'(z)/\sigma(z)$. The reason for this discrepancy is that here the function ψ incorporates the action of the backward shift.

It is important to recognize that the integral on the right-hand side of (8) does not necessarily represent an inner product in H^2, or even in $L^2(\partial\mathbb{D})$. As we have already observed, the function $\sum\psi(g\circ\sigma)$ may have a singularity in $\mathbb{D}$ and hence might not be analytic. Under certain circumstances, it does not even belong to $L^2(\partial\mathbb{D})$.

Before proceeding with the proof of this lemma, we need to consider a minor technical detail. For φ analytic in $\mathbb{D}$, it is not always the case that $\varphi^{-1}(\partial\mathbb{D}) = \partial\varphi^{-1}(\mathbb{D})$. These sets do coincide, however, whenever φ is analytic in a neighborhood of the closure of $\varphi^{-1}(\mathbb{D})$. Any rational map $\varphi:\mathbb{D}\to\mathbb{D}$ satisfies this condition, since it can only have a finite number of poles, none of which can belong to the closure of $\varphi^{-1}(\mathbb{D})$.

Proof of Lemma 3. Fix an appropriate set of branch cuts for φ^{-1} and consider a complete set of branches φ_1^{-1}, $\varphi_2^{-1},\dots,\varphi_N^{-1}$. Let P denote the set of all branch points for φ^{-1} that lie on $\partial\mathbb{D}$ and, for each j in $\{1,2,\dots,n\}$, define $\Gamma_j = \varphi_j^{-1}(\partial\mathbb{D}\setminus P)$. Note that the sets Γ_j are pairwise disjoint and that

$$\bigcup_{j=1}^{N}\Gamma_j = \varphi^{-1}(\partial\mathbb{D}\setminus P).$$

Because P is a finite set, we see that

$$\int_{\partial\varphi^{-1}(\mathbb{D})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = \int_{\varphi^{-1}(\partial\mathbb{D})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = \sum_{j=1}^{N}\int_{\Gamma_j} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta}. \tag{9}$$

Since φ is one-to-one on each Γ_j, we can apply the change of variables $\varphi(\zeta)=\eta$ to each of the integrals on the right-hand side of (9). Because $\zeta = \varphi_j^{-1}(\eta)$ for ζ in Γ_j, we obtain

$$\int_{\partial\varphi^{-1}(\mathbb{D})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = \int_{\partial\mathbb{D}} f(\eta)\sum_{j=1}^{N}\frac{\tilde{g}(\varphi_j^{-1}(\eta))}{\varphi_j^{-1}(\eta)\varphi'(\varphi_j^{-1}(\eta))}\,d\eta = \int_{\partial\mathbb{D}} f(\eta)\sum_{j=1}^{N}\frac{(\varphi_j^{-1})'(\eta)}{\varphi_j^{-1}(\eta)}\tilde{g}\big(\varphi_j^{-1}(\eta)\big)\,d\eta.$$

If η belongs to $\partial\mathbb{D}$, then $\tilde{g}(\varphi_j^{-1}(\eta)) = \overline{g(\sigma_j(\eta))}$ and $\varphi_j^{-1}(\eta) = \overline{\widetilde{\varphi_j^{-1}}(\eta)}$. One can easily check that $(\tilde{h})'(z) = (-1/z^2)\widetilde{(h')}(z)$ for all polynomials, and hence for any function that is analytic in a neighborhood of the point z. Thus a bit of computation shows that

$$\frac{\widetilde{(\varphi_j^{-1})'}(z)}{\widetilde{\varphi_j^{-1}}(z)} = \frac{z^2\sigma_j'(z)}{\sigma_j(z)}$$

whenever $1/\bar{z}$ is not a branch point for φ^{-1}. Combining all of these observations, we have

$$\int\limits_{\partial\varphi^{-1}(\mathbb{D})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = \int\limits_{\partial\mathbb{D}} f(\eta)\overline{\sum_{j=1}^{N}\frac{\eta^2\sigma_j'(\eta)}{\sigma_j(\eta)}g\big(\sigma_j(\eta)\big)}\,d\eta = \int\limits_{\partial\mathbb{D}} f(\eta)\overline{\sum_{j=1}^{N}\frac{\eta\sigma_j'(\eta)}{\sigma_j(\eta)}g\big(\sigma_j(\eta)\big)}\,\frac{d\eta}{\eta},$$

as we had hoped to show. □

One consequence of this lemma is that, as long as it belongs to $L^2(\partial\mathbb{D})$, the function $\sum\psi(g\circ\sigma)$ is not affected by the choice of branch cuts.

While the multiple-valued function φ^{-1} can be quite complicated in terms of its branch cuts, it can have only a single pole. In particular, a rational map has only one limit as $|z|$ goes to ∞. For the duration of this paper, we will write $\varphi(\infty)$ to denote this quantity; that is,

$$\varphi(\infty) = \lim_{|z|\to\infty}\varphi(z),$$

with the understanding that $\varphi(\infty)=\infty$ if no finite limit exists. The value of $\varphi(\infty)$ determines the basic geometric properties of the set $\varphi^{-1}(\mathbb{D})$. It is always the case that $\varphi^{-1}(\mathbb{D})$ is an open set containing $\mathbb{D}$. As long as $|\varphi(\infty)|\neq 1$, the boundary $\partial\varphi^{-1}(\mathbb{D})$ consists of a finite number of bounded curves. If $|\varphi(\infty)|>1$, then $\varphi^{-1}(\mathbb{D})$ is the union of bounded regions enclosed within $\partial\varphi^{-1}(\mathbb{D})$. On the other hand, if $|\varphi(\infty)|<1$, the set $\varphi^{-1}(\mathbb{D})$ is unbounded and has bounded complement. If $|\varphi(\infty)|=1$, then at least one component of $\partial\varphi^{-1}(\mathbb{D})$ must be unbounded, which means that both $\varphi^{-1}(\mathbb{D})$ and its complement are unbounded sets. (See Fig. 2.)

For the purposes of this paper, we will need to consider each of these cases separately. We start with the most straightforward situation.

Proposition 4. *Suppose that φ is a nonconstant rational map that takes $\mathbb{D}$ into itself, with $|\varphi(\infty)|>1$. Then*

$$\big(C_\varphi^* f\big)(z) = \sum\psi(z)f\big(\sigma(z)\big) + \frac{f(0)}{1-\overline{\varphi(\infty)}z}, \tag{10}$$

where σ and ψ are the multiple-valued functions defined in Lemma 3 *and the summation is taken over all branches of σ.*

Before giving the proof of this proposition, we need to discuss one aspect of its statement. Clearly the expression on the right-hand side of (10) is undefined at $z=1/\overline{\varphi(\infty)}$, both because that point is a singularity for $1/(1-\overline{\varphi(\infty)}z)$ and because σ, the denominator of ψ, vanishes there. This expression also has singularities at the finitely many points where σ' is undefined. It will follow from our subsequent discussion that all of these singularities are removable. A similar observation also pertains to the statements of Propositions 5 and 6.

Proof of Proposition 4. Since the polynomials are dense in H^2, it suffices to consider $\langle f, C_\varphi^*(g)\rangle$ for polynomials f and g. Observe that $\tilde{g}$, as defined in line (4), is analytic in $\{z\colon |z|>0\}$ and that $\tilde{g}$ agrees with $\bar{g}$ on the boundary of $\mathbb{D}$. Thus

$$\big\langle f, C_\varphi^*(g)\big\rangle = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} f\big(\varphi(\zeta)\big)\overline{g(\zeta)}\,\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta}. \tag{11}$$

We wish to transfer this last integral from $\partial\mathbb{D}$ to $\partial\varphi^{-1}(\mathbb{D})$.

Since $|\varphi(\infty)|>1$, we know that $\varphi^{-1}(\mathbb{D})$ is a bounded set, one component of which contains $\mathbb{D}$. First of all, suppose A is a component of $\varphi^{-1}(\mathbb{D})$ that does not contain $\mathbb{D}$. Since φ is analytic inside A and the point 0 does not belong to A, we see that

$$\int\limits_{\partial A} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = 0.$$

Now let B denote the component of $\varphi^{-1}(\mathbb{D})$ that contains $\mathbb{D}$. (See Fig. 3.) If B is not simply connected, we can draw a set of nonintersecting contours (avoiding the origin) that connect the exterior boundary of B to the boundary of each

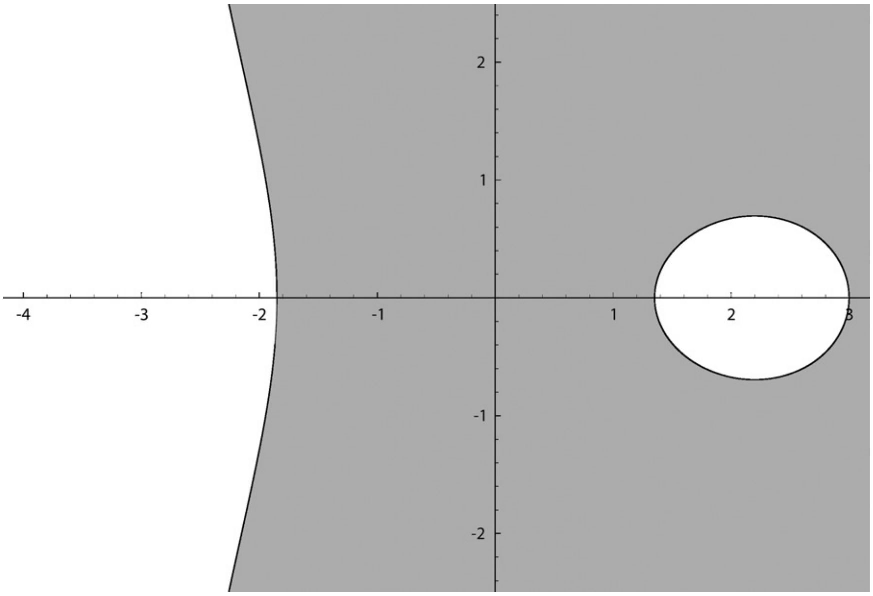

Fig. 2. The set $\varphi^{-1}(\mathbb{D})$ for $\varphi(z) = (z^2 - 1)/(z^2 + z - 4)$.

of the holes inside B. In this way, traversing every such contour in both directions, we can regard ∂B as being a single connected curve. Since φ is analytic inside B, it is possible to deform $\partial\mathbb{D}$ to ∂B without passing over any singularities of φ and without passing over 0. Thus we can apply Cauchy's theorem and Eq. (11) to see that

$$\langle f, C_\varphi^*(g)\rangle = \frac{1}{2\pi i}\int\limits_{\partial B} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{\partial\varphi^{-1}(\mathbb{D})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta}.$$

Hence Lemma 3 dictates that

$$\langle f, C_\varphi^*(g)\rangle = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} f(\zeta)\overline{\sum \psi(\zeta) g\big(\sigma(\zeta)\big)}\,\frac{d\zeta}{\zeta} \tag{12}$$

for any polynomials f and g.

Consider the function $\sum \psi(g\circ\sigma)$ for a fixed polynomial g. We would like to show that $\sum \psi(g\circ\sigma)$ belongs to $L^2(\partial\mathbb{D})$. To that end, let us examine its Fourier series with respect to the basis $\{z^n\}_{n=-\infty}^{\infty}$. Since line (12) holds for any polynomial f, we only need to compute the Fourier coefficients corresponding to negative powers of z. Let n be a natural number and take $f(z) = z^{-n}$. Lemma 3 dictates that

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} f(\zeta)\overline{\sum \psi(\zeta) g\big(\sigma(\zeta)\big)}\,\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{\partial\varphi^{-1}(\mathbb{D})} \frac{\tilde{g}(\zeta)}{\varphi(\zeta)^n}\,\frac{d\zeta}{\zeta}.$$

Since φ is a rational map, the only singularities of the integrand on the right-hand side occur when $\zeta = 0$ or when $\varphi(\zeta) = 0$; that is, within $\varphi^{-1}(\mathbb{D})$. Recall that the set $\varphi^{-1}(\mathbb{D})$ is bounded. Therefore, for R sufficiently large, Cauchy's theorem shows that

$$\frac{1}{2\pi i}\int\limits_{\partial\varphi^{-1}(\mathbb{D})} \frac{\tilde{g}(\zeta)}{\varphi(\zeta)^n}\,\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{\mathcal{C}_R} \frac{\tilde{g}(\zeta)}{\varphi(\zeta)^n}\,\frac{d\zeta}{\zeta},$$

where $\mathcal{C}_R$ denotes the circle of radius R centered at the origin. Letting R go to ∞, we see that

$$\frac{1}{2\pi i}\int\limits_{\mathcal{C}_R} \frac{\tilde{g}(\zeta)}{\varphi(\zeta)^n}\,\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \frac{\tilde{g}(R\zeta)}{\varphi(R\zeta)^n}\,\frac{d\zeta}{\zeta} \to \frac{\overline{g(0)}}{\varphi(\infty)^n}.$$

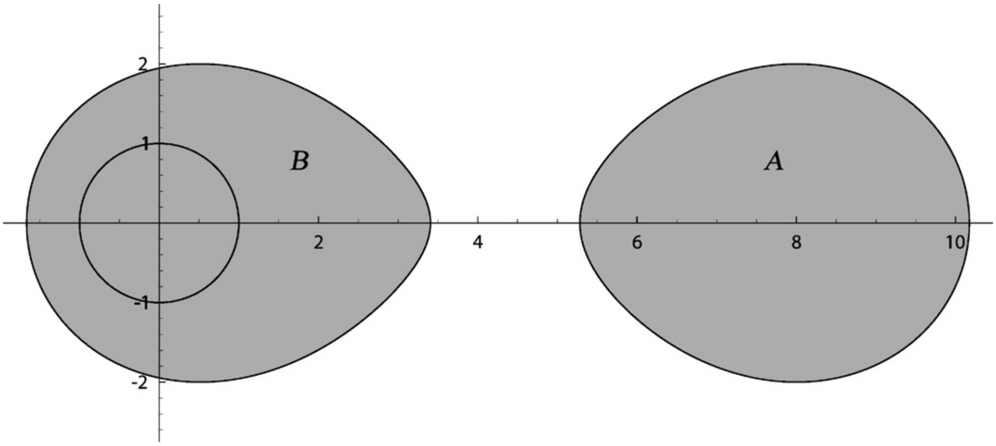

Fig. 3. The set $\varphi^{-1}(\mathbb{D})$ for $\varphi(z) = z^2/17 - z/2$, along with $\partial\mathbb{D}$.

In other words, the $(-n)$th Fourier coefficient of $\sum \psi(g \circ \sigma)$ equals $g(0)/\overline{\varphi(\infty)^n}$, where we understand this quantity to be 0 if $\varphi(\infty) = \infty$. Therefore, if $f(z) = \sum_{n=-N}^{N} a_n z^n$, we see that

$$\left|\frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\overline{\sum \psi(\zeta)g\bigl(\sigma(\zeta)\bigr)}\,\frac{d\zeta}{\zeta}\right| = \left|\sum_{n=1}^{N} a_{-n}\frac{\overline{g(0)}}{\varphi(\infty)^n} + \left\langle \sum_{n=0}^{N} a_n z^n, C_\varphi^*(g)\right\rangle\right|$$

$$\leqslant \|f\|_2\left(\sqrt{\frac{|g(0)|^2}{|\varphi(\infty)|^2 - 1}} + \|C_\varphi\|\|g\|_2\right). \tag{13}$$

Consequently $\sum \psi(g \circ \sigma)$ must belong to $L^2(\partial\mathbb{D})$.

In view of line (12), we see that $C_\varphi^*(g)$ is simply the projection of $\sum \psi(g \circ \sigma)$ onto H^2. Note that

$$\sum_{n=1}^{\infty} \frac{g(0)}{\overline{\varphi(\infty)^n}}\frac{1}{z^n} = \frac{g(0)(\overline{\varphi(\infty)}z)^{-1}}{1 - (\overline{\varphi(\infty)}z)^{-1}} = \frac{g(0)}{\overline{\varphi(\infty)}z - 1}.$$

(For the purposes of this calculation, we can assume that z belongs to $\partial\mathbb{D}$.) Consequently the projection of $\sum \psi(g \circ \sigma)$ onto H^2 equals

$$\sum \psi(z)g\bigl(\sigma(z)\bigr) + \frac{g(0)}{1 - \overline{\varphi(\infty)}z},$$

which is precisely $C_\varphi^*(g)$. □

Next we shall consider the situation when φ^{-1} has a pole inside the disk; that is, when $|\varphi(\infty)| < 1$. The ideas underlying our calculations will be similar to what we have already seen, although $\varphi^{-1}(\mathbb{D})$ is an unbounded set and $\partial\mathbb{D}$ is not enclosed within $\partial\varphi^{-1}(\mathbb{D})$. Hence, as discussed in Example 2, we cannot use Cauchy's theorem in the same manner as in the previous proof, since in deforming $\partial\mathbb{D}$ to $\partial\varphi^{-1}(\mathbb{D})$ we would pass over the point 0 and possibly some poles of φ. Nevertheless, the result we obtain looks quite familiar.

Proposition 5. *Suppose that φ is a nonconstant rational map that takes $\mathbb{D}$ into itself, with $|\varphi(\infty)| < 1$. Then*

$$\bigl(C_\varphi^* f\bigr)(z) = \sum \psi(z) f\bigl(\sigma(z)\bigr) + \frac{f(0)}{1 - \overline{\varphi(\infty)}z},$$

where σ and ψ are the multiple-valued functions defined in Lemma 3 *and the summation is taken over all branches of σ.*

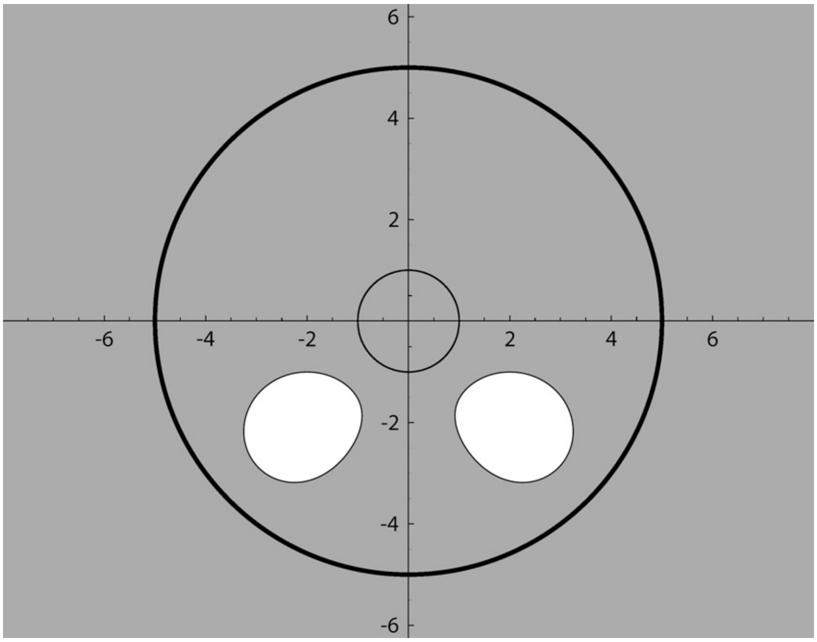

Fig. 4. The set $\varphi^{-1}(\mathbb{D})$ for $\varphi(z) = (z^2 + 2iz + 3)/(2z^2 + 8iz - 14)$, along with $\partial\mathbb{D}$ and $\mathcal{C}_R$ for $R = 5$.

Proof. Let f and g be polynomials, and consider

$$\langle f, C_\varphi^*(g)\rangle = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta}.$$

Because $\partial\varphi^{-1}(\mathbb{D})$ is bounded, we can find R sufficiently large so that $\mathcal{C}_R$, the circle of radius R centered at the origin, surrounds $\partial\varphi^{-1}(\mathbb{D})$. (See Fig. 4.) Since $\partial\varphi^{-1}(\mathbb{D})$ encloses $\varphi^{-1}(\mathbb{C}\setminus\overline{\mathbb{D}})$, the map φ is analytic in the region outside $\partial\varphi^{-1}(\mathbb{D})$. Hence Cauchy's theorem dictates that

$$\int_{\mathcal{C}_R} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = \int_{\partial\mathbb{D}} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} + \int_{\partial\varphi^{-1}(\mathbb{C}\setminus\overline{\mathbb{D}})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta},$$

so that

$$\begin{aligned}\langle f, C_\varphi^*(g)\rangle &= -\frac{1}{2\pi i}\int_{\partial\varphi^{-1}(\mathbb{C}\setminus\overline{\mathbb{D}})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\mathcal{C}_R} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta}\\ &= \frac{1}{2\pi i}\int_{\partial\varphi^{-1}(\mathbb{D})} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\mathcal{C}_R} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta}.\end{aligned}$$

It follows from Lemma 3 that

$$\langle f, C_\varphi^*(g)\rangle = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\overline{\sum\psi(\zeta)g\big(\sigma(\zeta)\big)}\,\frac{d\zeta}{\zeta} + \frac{1}{2\pi i}\int_{\mathcal{C}_R} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta}.$$

If we let R go to ∞, we see that

$$\frac{1}{2\pi i}\int_{\mathcal{C}_R} f\big(\varphi(\zeta)\big)\tilde{g}(\zeta)\,\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f\big(\varphi(R\zeta)\big)\tilde{g}(R\zeta)\,\frac{d\zeta}{\zeta} \to f\big(\varphi(\infty)\big)\overline{g(0)}.$$

(Recall our assumption that $|\varphi(\infty)| < 1$.) Notice that we can rewrite

$$f\big(\varphi(\infty)\big)\overline{g(0)} = \big\langle C_{\varphi(\infty)}(f), C_0(g)\big\rangle,$$

where $C_{\varphi(\infty)}$ and C_0 can simply be understood as point-evaluation functionals. Theorem 1 tells us that $\langle C_{\varphi(\infty)}(f), C_0(g)\rangle = \langle f, T_\chi C_0(g)\rangle$, where

$$\chi(z) = \frac{1}{1-\overline{\varphi(\infty)}z}.$$

In other words,

$$\langle f, C_\varphi^*(g)\rangle = \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} f(\zeta)\Big(\overline{\sum \psi(\zeta)g\big(\sigma(\zeta)\big) + \chi(\zeta)g(0)}\Big)\frac{d\zeta}{\zeta}$$

for all polynomials f and g.

As in the proof of Proposition 4, let us consider the Fourier series for the function $\sum \psi(g\circ\sigma)$. If $f(z) = z^{-n}$, Lemma 3 dictates that

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} f(\zeta)\overline{\sum \psi(\zeta)g\big(\sigma(\zeta)\big)}\frac{d\zeta}{\zeta} = \frac{1}{2\pi i}\int\limits_{\partial\varphi^{-1}(\mathbb{D})} \frac{\tilde{g}(\zeta)}{\varphi(\zeta)^n}\frac{d\zeta}{\zeta}.$$

The only singularities of the integrand on the right-hand side occur within the set $\varphi^{-1}(\mathbb{D})$. As we have already noted, the curves that make up $\partial\varphi^{-1}(\mathbb{D})$ enclose the preimage of the complement of the closed disk. In other words, this integral equals 0 for all n, which means that the Fourier coefficients of $\sum\psi(g\circ\sigma)$ corresponding to negative powers of z are all 0. Hence an argument similar to that of line (13) shows that $\sum\psi(g\circ\sigma)$ belongs to $L^2(\partial\mathbb{D})$, and in fact to H^2. Therefore $C_\varphi^*(g)$ is the projection of $\sum\psi(g\circ\sigma) + \chi g(0)$ onto the Hardy space. It is clear that χ belongs to H^2, so we obtain the desired result. □

At first glance, it may appear that Propositions 4 and 5 are saying the same thing, but in fact there is a subtle difference. As noted in the proof of Proposition 5, both of the functions $\sum\psi(f\circ\sigma)$ and χ belong to H^2. Thus, when $|\varphi(\infty)| < 1$, we can write

$$C_\varphi^* = W_{\psi,\sigma} + T_\chi C_0,$$

where all of these operators actually take H^2 into itself.

Finally we turn our attention to the case where $|\varphi(\infty)| = 1$. In this situation, as we have already mentioned, the geometric properties of $\varphi^{-1}(\mathbb{D})$ can be rather complicated. Furthermore, as we can deduce from the next proposition, the function $\sum\psi(z)f(\sigma(z))$ does not belong to $L^2(\partial\mathbb{D})$. Rather than dealing with this case on its own terms, we will obtain the desired result as a consequence of Proposition 5.

Proposition 6. *Suppose that φ is a nonconstant rational map that takes $\mathbb{D}$ into itself, with $|\varphi(\infty)| = 1$. Then*

$$\big(C_\varphi^* f\big)(z) = \sum \psi(z)f\big(\sigma(z)\big) + \frac{f(0)}{1-\overline{\varphi(\infty)}z},$$

where σ and ψ are the multiple-valued functions defined in Lemma 3 and the summation is taken over all branches of σ.

Proof. Fix a real number r, with $0 < r < 1$, and consider the map $r\varphi(z)$. Observe that $|r\varphi(\infty)| < 1$, so Proposition 5 applies to the operator $C_{r\varphi}^*$. Define σ and ψ in terms of the original map φ; we shall state the formula for $C_{r\varphi}^*$ with respect to these functions. Since $(r\varphi)^{-1}(z) = \varphi^{-1}(z/r)$, it follows that

$$\frac{1}{\overline{(r\varphi)^{-1}(1/\bar{z})}} = \frac{1}{\overline{\varphi^{-1}(1/\overline{rz})}} = \sigma(rz)$$

and

$$\frac{z(\sigma(rz))'}{\sigma(rz)} = \frac{(rz)\sigma'(rz)}{\sigma(rz)} = \psi(rz).$$

Consequently

$$\left(C_{r\varphi}^* f\right)(z) = \sum \psi(rz) f\left(\sigma(rz)\right) + \frac{f(0)}{1 - \overline{\varphi(\infty)} rz}. \tag{14}$$

Observe that $C_{r\varphi} = C_\varphi C_\rho$, where $\rho(z) = rz$. Since C_ρ is self-adjoint (as we can see from Theorem 1), it follows that

$$\left(C_{r\varphi}^* f\right)(z) = \left(C_\rho^* C_\varphi^* f\right)(z) = \left(C_\rho C_\varphi^* f\right)(z) = \left(C_\varphi^* f\right)(rz).$$

Combining this observation with line (14), we see that

$$\left(C_\varphi^* f\right)(rz) = \sum \psi(rz) f\left(\sigma(rz)\right) + \frac{f(0)}{1 - \overline{\varphi(\infty)} rz}$$

for all z in $\mathbb{D}$. Making the substitution $\zeta = rz$, we conclude that

$$\left(C_\varphi^* f\right)(\zeta) = \sum \psi(\zeta) f\left(\sigma(\zeta)\right) + \frac{f(0)}{1 - \overline{\varphi(\infty)} \zeta}$$

whenever $|\zeta| < r$. Since r is arbitrary, our assertion follows. $\square$

Combining Propositions 4, 5, and 6, we can now state our main result.

Theorem 7. *Suppose that φ is a nonconstant rational map that takes $\mathbb{D}$ into itself. Then*

$$\left(C_\varphi^* f\right)(z) = \sum \psi(z) f\left(\sigma(z)\right) + \frac{f(0)}{1 - \overline{\varphi(\infty)} z},$$

where

$$\begin{aligned} \sigma(z) &= 1/\widetilde{\varphi^{-1}}(z) = 1/\overline{\varphi^{-1}(1/\bar{z})}, \\ \psi(z) &= \frac{z\sigma'(z)}{\sigma(z)}, \\ \varphi(\infty) &= \lim_{|z| \to \infty} \varphi(z), \end{aligned}$$

and the summation is taken over all branches of σ.

Notice that this result agrees exactly with line (2), the adjoint formula for composition operators with linear fractional symbol. Furthermore, we obtain the following corollary.

Corollary 8. *Suppose that φ is a rational map that takes $\mathbb{D}$ into itself. If $\varphi(\infty) = \infty$, then C_φ^* is a (multiple-valued) weighted composition operator.*

5. Further examples

Prior to this paper, there were a few concrete examples of rational φ (in addition to the linear fractional maps) for which the adjoint C_φ^* could be described precisely. We will conclude by considering several such results in the context of Theorem 7.

Example 3. Let $\varphi(z) = z^m$ for some natural number m. It is easy to calculate C_φ^* simply by considering its action on the orthonormal basis $\{z^n\}_{n=0}^\infty$. (See Exercise 9.1.1 in [4].) Applying Theorem 7, we see that

$$\left(C_\varphi^* f\right)(z) = \sum_{j=1}^m \frac{f(\sigma_j(z))}{m},$$

there $\sigma_1, \sigma_2, \ldots, \sigma_m$ constitute all the branches of the function $\sqrt[m]{z}$. This result agrees with the formulas previously stated in [8] and [9].

Example 4. The one specific example discussed by Cowen and Gallardo-Gutiérrez [3] is the map $\varphi(z) = (z^2 + z)/2$. Let us consider a slightly more general case

$$\varphi(z) = az^2 + bz,$$

with $a \neq 0$. Note that $\varphi(\infty) = \infty$, so C_φ^* is actually a multiple-valued weighted composition operator. It is easy to show that

$$\sigma(z) = \frac{\bar{b}z \pm \sqrt{\bar{b}^2 z^2 + 4\bar{a}z}}{2}$$

and

$$\psi(z) = \frac{z\sigma'(z)}{\sigma(z)} = \frac{1}{2}\left(1 \pm \frac{\bar{b}\sqrt{\bar{b}^2 z^2 + 4\bar{a}z}}{\bar{b}^2 z + 4\bar{a}}\right).$$

Therefore Theorem 7 says that

$$\left(C_\varphi^* f\right)(z) = \sum_{j=1}^{2} \frac{1}{2}\left(1 + (-1)^j \frac{\bar{b}\sqrt{\bar{b}^2 z^2 + 4\bar{a}z}}{\bar{b}^2 z + 4\bar{a}}\right) f\left(\frac{\bar{b}z + (-1)^j\sqrt{\bar{b}^2 z^2 + 4\bar{a}z}}{2}\right).$$

When $a = b = 1/2$, this expression agrees with the calculation found in [3].

Example 5. As we mentioned earlier, several years ago Bourdon [1] calculated the adjoint C_φ^* for φ belonging to a particular class of rational maps. One example he considered was

$$\varphi(z) = \frac{z^2 - 6z + 9}{z^2 - 10z + 13}.$$

Note that

$$\sigma(z) = \frac{3z - 5 \pm 2\sqrt{3 - 2z}}{9z - 13}$$

and

$$\psi(z) = \frac{\pm 2z}{\sqrt{3 - 2z}(3z - 4 \pm \sqrt{3 - 2z})}.$$

Therefore $(C_\varphi^* f)(z)$ equals

$$\sum_{j=1}^{2} \frac{(-1)^j 2z}{\sqrt{3 - 2z}(3z - 4 + (-1)^j\sqrt{3 - 2z})} f\left(\frac{3z - 5 + (-1)^j 2\sqrt{3 - 2z}}{9z - 13}\right) + \frac{f(0)}{1 - z}.$$

While the two formulas look somewhat different, it is not difficult to show that this result is identical to Bourdon's.

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